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1979 J. Phys. A: Math. Gen. 12 L47

(http://iopscience.iop.org/0305-4470/12/2/001)

J. Phys. A: Math. Gen., Vol. 12, No. 2, 1979. Printed in Great Britain

LETTER TO THE EDITOR

Comments on the field-theoretic formulation of the Yang–Lee edge singularity

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Received 31 October 1978

Abstract. Fisher has recently shown that the character of the Yang–Lee edge singularity may be obtained from a one-component field theory with pure imaginary ϕ^3 coupling. We point out that ϕ^4 interactions are not required to stabilise this theory and that, even if they are present, their contribution is likely to be insignificant compared with that from the ϕ^3 interactions, as the dimension of space is lowered through four.

There has been recent interest in the problem of the character of the Yang–Lee zeros on the imaginary field axis for the ferromagnetic Ising model (Fisher 1978). These zeros arise in the following way (Yang and Lee 1952). Suppose we consider a system of N Ising spins; then the free energy F is given by the expression

$$\exp(-NF/kT) = P_N(Z)\exp(NH/kT) \tag{1}$$

where H is the magnetic field and $Z = \exp(-2H/kT)$. $P_N(Z)$ is a polynomial of degree N in Z and the coefficient of Z^n is the contribution to the partition function of the configuration with the number of 'down spins' equal to n. The zeros of this polynomial are called the Yang–Lee zeros, and if as $N \to \infty$, for some temperature T_c, the zeros close in onto the real Z axis, then there is a phase transition for this value of Z (Yang and Lee 1952). In particular, for the ferromagnetic case it can be shown that (Lee and Yang 1952) the zeros lie on the unit circle in the Z plane, that is the imaginary axis in the H plane. In the limit $N \to \infty$ the distribution of zeros on the unit circle $Z = e^{i\theta}$ can be described in terms of a density function $g(\theta)$ so that

$$\sum_{n=0}^{\infty} \to N\int_0^{2\pi} g(\theta)\,d\theta. \tag{2}$$

The isotherms in the M–H diagram can be calculated from this function. Conversely, from the M–H diagram the form of $g(\theta)$ can be inferred; in particular, the form of $g(\theta)$ for $T > T_c$ calculated from high-temperature and high-field expansions is (Kortmann and Griffiths 1971)

$$\begin{aligned} g(\theta) &= 0 \qquad & |\theta| < \theta_g \\ g(\theta) &\simeq (|\theta| - \theta_g)^\sigma \qquad & |\theta| > \theta_g. \end{aligned} \tag{3}$$

This form of the density function can be described by a branch cut along the imaginary field axis in the H plane with a gap between $\pm\frac{1}{2}i\theta_g kT$. The edges of this gap are the Yang–Lee edge singularities which are characterised by the exponent σ. If, following

Fisher (1978), we assume a density function of the above form then

$$m = h^{\sigma} \tag{4}$$

where

$$m = M - M(\mathrm{i}H_0, T)$$

$$h = H - \mathrm{i}H_0$$

and

$$H_0 = \tfrac{1}{2}\theta_g kT.$$

The susceptibility is therefore divergent as $h \to 0$ if $\sigma < 1$ since

$$\chi = \partial M/\partial H \sim h^{\sigma-1}. \tag{5}$$

We therefore need a field theory for the Ising model which has a divergent susceptibility as $h \to 0$ in order to calculate σ. A suitable Hamiltonian density is given by Fisher (1978):

$$\mathcal{H} = \tfrac{1}{2}(\nabla\phi)^2 + (\mathrm{i}g_0/3!)\phi^3 + (u_0/4!)\phi^4. \tag{6}$$

This can be constructed from the standard Hamiltonian density for an Ising-like model by applying a translation to the field and then constraining the coefficient of ϕ^2 to be zero, giving a divergent suceptibility. Naive dimensional analysis gives $g_0 = \mu^{(6-d)/2}g$ and $u_0 = \mu^{4-d}u$, where d is the dimension of space, μ is a momentum scale and g, u are dimensionless. Thus near $d = 6$ it is clear that the ϕ^3 term dominates and we consider a Hamiltonian density of the form

$$\mathcal{H} = \tfrac{1}{2}(\nabla\phi)^2 + (\mathrm{i}g_0/3!)\phi^3. \tag{7}$$

The value of σ obtained from this theory as an expansion in $\epsilon = 6 - d$ is (Fisher 1978, Amit 1976)

$$\sigma = \tfrac{1}{2} - \tfrac{1}{12}\epsilon - \tfrac{79}{3888}\epsilon^2 + \mathrm{O}(\epsilon^3). \tag{8}$$

In order that the ϵ expansion exists, the Hamiltonian density (7) must give rise to a well defined theory. We will show that this is a consequence of the imaginary coupling constant. If we consider a ϕ^3 theory with real coupling constant then the Hamiltonian is not bounded below, and therefore tunnelling from the classical ground state $\phi = 0$ to the region where ϕ is large and positive is possible. This tunnelling is described by the instanton solutions (Brézin 1977, McKane 1978) and is reflected in the fact that the asymptotic perturbation series has non-oscillatory terms and so resummation techniques cannot be applied. This can also be seen by noticing that the perturbation expansion is in g^2; therefore, neglecting the subtractions introduced by renormalisation, the series will not oscillate. However, if we consider an imaginary coupling constant as in (7), then there are no non-trivial real instanton solutions, the perturbation series is oscillatory and one may hope to resum it uniquely by, e.g., Padé–Borel methods. In fact, standard instanton techniques (Brézin 1977, Houghton *et al* 1978) show that at high order the ϵ-expansion (8) behaves like $\epsilon^K a^K K! \ldots$ where $a = -\frac{5}{18}$. Unfortunately, in this case this asymptotic oscillatory nature of the ϵ expansion is not evident in the low-order terms (8) and resummation is not justifiable with such a short series. In summary, the theory defined by the Hamiltonian density (7) appears to be stable in the absence of ϕ^4 interactions. The existence of pure imaginary coupling plays a similar role in the field theory for the Reggeon calculus (Cardy 1977).

We have shown that the ϕ^4 terms are not required to make the theory stable, but they are certainly present as indicated in (6) and we wish to show that their presence does not change the predicted behaviour (4) at the singularity significantly. According to the discussion following equation (6), if the dimension of space d is close to 6, the ϕ^4 interactions are irrelevant. However, we wish to check that this is so for d close to 4 (where they become relevant for the free-field fixed point $g^* = 0$).

In order to consider the effect of a ϕ^4 interaction we follow the method of Amit *et al* (1977) and calculate the anomalous dimensions of terms of this form. Similar calculations for the Edward–Anderson model for a spin glass can be found in Elderfield and McKane (1978). In particular, we consider the linearly-independent operators $A_1 \ldots A_7$ which have the same naive dimension as ϕ^4 and so must be renormalised simultaneously (Brézin *et al* 1976):

$$A_1(x) = \phi^4(x)/4! \tag{9a}$$

$$A_2(x) = -\mu^{-\epsilon/2}\phi^2(x)\Box\phi(x)/3! \tag{9b}$$

$$A_3(x) = \mu^{-\epsilon}[\Box\phi(x)]^2/2! \tag{9c}$$

$$A_4(x) = \mu^{-\epsilon}\phi(x)\Box^2\phi(x)/2! \tag{9d}$$

$$A_5(x) = \mu^{-\epsilon/2}\Box[\phi^3(x)]/3! \tag{9e}$$

$$A_6(x) = \mu^{-\epsilon}\Box[\phi(x)\Box\phi(x)]/2! \tag{9f}$$

$$A_7(x) = \mu^{-\epsilon}\Box^2[\phi^2(x)]/2! \tag{9g}$$

where $\Box$ denotes $\nabla_i\nabla_i$.

We then define vertex functions $\Gamma_\alpha^{\{a\}}$ in the following way:

$$\Gamma_\alpha^{\{1\}} = \Gamma_\alpha^{(4)} \tag{10a}$$

$$\Gamma_\alpha^{\{2\}} = [(\partial/\partial q_1^2) + (\partial/\partial q_2^2) + (\partial/\partial q_3^2)]\Gamma_\alpha^{(3)} \tag{10b}$$

$$\Gamma_\alpha^{\{3\}} = [(\partial/\partial q_1^2)(\partial/\partial q_2^2)]\Gamma_\alpha^{(2)} \tag{10c}$$

$$\Gamma_\alpha^{\{4\}} = [(\partial/\partial q_1^4) + (\partial/\partial q_2^4)]\Gamma_\alpha^{(2)} \tag{10d}$$

where $\Gamma_\alpha^{(n)}$ is the n-point vertex function with an insertion of type α, and q_1, q_2, q_3 are the external momenta. We then define our renormalised vertex function as

$$\hat{\Gamma}_{\mathrm{R}a}^{\{c\}} = Z_{ab}Z^{l_c/2}\mu^{-\epsilon(l_c-4)/2}\Gamma_b^{\{c\}} = Z_{ab}Z^{l_c/2}\hat{\Gamma}_b^{\{c\}} \tag{11}$$

where l_c is the number of legs associated with a vertex function of type $\{c\}$, Z is the usual wavefunction renormalisation constant and Z_{ab} is a matrix determined by imposing the normalisation condition

$$\hat{\Gamma}_{\mathrm{R}a}^{\{b\}}|_\mu = \delta_a^b \tag{12}$$

where μ is the symmetry point appropriate to the vertex type. These definitions lead to a renormalisation group equation

$$\{[(\mu\partial/\partial\mu) + (\beta(u)\partial/\partial u) - \tfrac{1}{2}l_d\eta(u) - \tfrac{1}{2}\epsilon(4-l_d)]\delta_{ab} - \gamma_{ab}(u)\}\hat{\Gamma}_{\mathrm{R}b}^{\{d\}} = 0 \tag{13}$$

where

$$\gamma_{ab} = (-Z_{ac}\mu\partial/\partial\mu)|_g Z_{cb}^{-1} + \tfrac{1}{2}\epsilon\sum_c Z_{ac}(l_c-4)Z_{cb}^{-1}. \tag{14}$$

The anomalous dimensions we wish to calculate are the eigenvalues λ_α of γ_{ab}. We can see the importance of these λ_α by calculating the dimensions of $\Gamma^{(4)}$ and $\Gamma_\alpha^{\{1\}}$ from the corresponding renormalisation group equations and then obtaining the condition that as the momentum scale $k \to 0$, $\Gamma_\alpha^{\{1\}}$ vanishes faster than $\Gamma^{(4)}$, that is the combination of insertions corresponding to λ_α is irrelevant. The condition we obtain is

$$2-\epsilon-\lambda_\alpha>0 \tag{15}$$

(Amit *et al* 1977). We are particularly interested in the relevance of the ϕ^4 operators near $d=4$, where we would expect from naive dimensional analysis that they might be important; in this case condition (15) becomes

$$\lambda_\alpha<0. \tag{16}$$

The calculation of the matrix γ_{ab} for general n and a traceless ϕ^3 interaction is given in Amit *et al* (1977). We want to calculate it for $n=1$ when the interaction is not traceless. This means that our results are slightly different from those that arise by taking $n=1$ in the general result.

The insertions A_5–A_7 do not need to be considered as they involve total derivatives, and so give rise to convergent terms in (10). We thus calculate $\hat{\Gamma}_a^{\{b\}}|_\mu$ for $a, b = 1, 2, 3, 4$ and the graphs that give rise to non-zero contributions are shown in figure 1. As all off-diagonal elements of $\hat{\Gamma}^{\{4\}}$ are zero and

$$Z_{ab}^{-1} = Z^{l_b/2}\hat{\Gamma}_a^{\{b\}} \tag{17}$$

from the normalisation condition (12), then from the definition of γ_{ab} (14) we see that all off-diagonal elements of the 4th column of γ_{ab} are zero. This means that the eigenvalue corresponding to an insertion of type four separates out. Its value calculated at the fixed point $K_d g^{*2} = \frac{2}{3}\epsilon$ was found to be $\lambda = -\frac{4}{9}\epsilon$, which satisfies the condition (16).

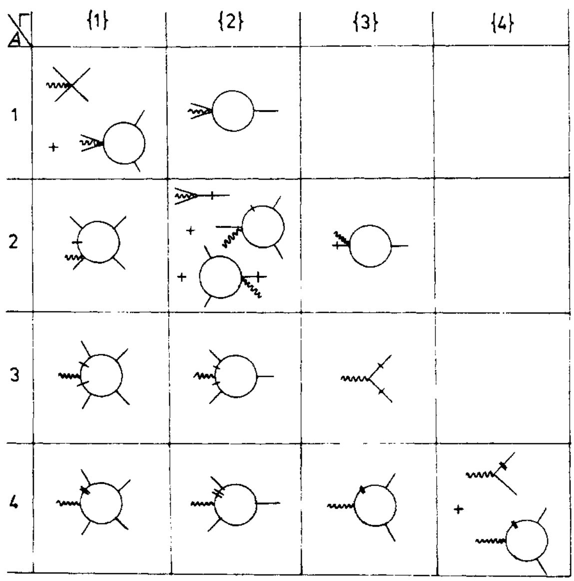

Figure 1. The graphs contributing to each vertex function $\Gamma^{\{a\}}$ for each insertion A_1–A_4. An extra stroke on a leg indicates a factor of q^2.

The matrix γ_{ab} for the first three operators is

$$\gamma_{ab}=\begin{bmatrix} \frac{17}{3}v^2K_d & \frac{1}{2}vK_d & 0 \\ -8v^3K_d & \frac{1}{12}(-6\epsilon+v^2K_d) & \frac{1}{9}vK_d \\ 12v^4K_d & v^3K_d & -\frac{1}{6}v^2K_d-\epsilon \end{bmatrix} \tag{18}$$

where $v=\mathrm{i}g_0\mu^{-\epsilon/2}$ is the dimensionless coupling constant and $K_d=S_d/(2\pi)^d$; S_d is the surface area of a unit sphere in d dimensions. To find the eigenvalues of γ_{ab} we look at $\det(\gamma-\lambda I)$ at the fixed point and notice that by adding $\frac{3}{2}v\times\mathrm{row}(2)$ to row(3), one eigenvalue, namely $\lambda=-\epsilon$, falls out immediately. The other eigenvalues we obtain are $\lambda=-\frac{28}{9}\epsilon$ and $\lambda=-\frac{10}{9}\epsilon$. Clearly all these eigenvalues satisfy condition (16) and therefore the anomalous dimensions of the ϕ^4 interaction indicate that it remains irrelevant even when the dimension of space is close to four.

In conclusion, we see that the Hamiltonian density (7) is well defined because the coupling constant is imaginary, and therefore the ϕ^4 terms are not required for stability; further, if a ϕ^4 interaction is included, it gives negligible contributions to the form of the singularity at the Yang–Lee edge.

We are grateful to M E Fisher for sending a copy of his work before publication. J E Kirkham thanks the Science Research Council for the award of a research studentship.

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